Some two-dimensional magneto-fluid-dynamic flows at low magnetic Reynolds number

By J. H. HORLOCK

Department of Mechanical Engineering, University of Liverpool

(Received 19 April 1962 and in revised form 18 September 1962)

A study is made of some two-dimensional flows of an inviscid, uniformly conducting fluid in the presence of applied magnetic fields, for the case in which the magnetic Reynolds number (the ratio of induced fields to applied fields) is small but the magnetic force coefficient (the ratio of electromagnetic forces to inertia forces) is of order unity.

Perturbations of uniform incompressible and compressible flows through aligned and crossed fields are considered. All these perturbations are strongly dependent upon the magnitude of the magnetic force coefficient. An experimental analogy of the flow through an aligned field is described.

1. Introduction

Early work in magnetohydrodynamics, notably that of Hartmann (1937), was directed towards the study of the changes in incompressible viscous flows produced by electromagnetic forces. The prospects for magnetogasdynamics, the fluid mechanics of ionized gases moving through magnetic fields, have recently been reviewed by Resler & Sears (1958*a*) and Shercliff (1959).

Among the dimensionless groups important in inviscid magneto-fluid-dynamic flow are (i) the magnetic Reynolds number R_m, which expresses the ratio of the induced flux density to the applied flux density, and (ii) the magnetic force coefficient S, which is the ratio of the electromagnetic force per unit volume to the inertia forces per unit volume in the fluid.

One-dimensional magnetogasdynamic flows, with crossed fields, have been studied by Resler & Sears (1958*b*). Two-dimensional perturbation flows at infinite, and at large but finite, magnetic Reynolds number have been discussed by Sears & Resler (1959), Resler & McCune (1960) and McCune (1960). Williams (1960) has obtained more general solutions for finite magnetic Reynolds number and finite magnetic force coefficient.

Hains, Yohler & Ehlers (1960) and Ludford (1961) have considered magneto-fluid-dynamic flows in which the magnetic Reynolds number is very small, and the applied magnetic field remains unperturbed by currents produced in the fluid. Hains *et al.* have described a general method of characteristics for use in supersonic flow under these conditions; they have also linearized the equations by considering small values of the magnetic force coefficient, and obtained solutions for both subsonic and supersonic flow. Ludford has studied the case in which, although R_m is small, the magnetic force coefficient is very large and only pressure and magnetic forces are present.

The present study is also confined to flows in which the applied field remains unchanged by current flows in the fluid ($R_m \ll 1$), but the force coefficient S is of order unity and the equations of motion are linearized by considering two-dimensional perturbations of a uniform flow. Viscous effects, Hall currents and ion slip are all neglected.

2. General equations of motion

Various authors (e.g. Resler & Sears 1958*a*) have given the basic equations for the motion of conducting fluids. For steady inviscid flow, the momentum equation is

$$\frac{1}{\rho}\nabla p + (\mathbf{q}.\nabla)\mathbf{q} = \frac{\mu(\mathbf{J}\times\mathbf{H})}{\rho}, \tag{1}$$

where $\mathbf{q}$ is the velocity vector, ρ the fluid density, p the pressure, $\mathbf{J}$ the current-density vector, μ the fluid permeability and $\mathbf{H}$ the field strength. The equation of continuity is

$$\nabla.(\rho\mathbf{q}) = 0. \tag{2}$$

Ohm's law requires that

$$\mathbf{J} = \sigma(\mathbf{E}+\mu\mathbf{q}\times\mathbf{H}), \tag{3}$$

where $\mathbf{E}$ is the electric field and σ the fluid conductivity, assumed uniform.

Maxwell's equations are

$$\nabla\times\mathbf{E} = 0, \tag{4}$$

and

$$\nabla\times\mathbf{H} = \mathbf{J}. \tag{5}$$

It follows from (5) that

$$\nabla.\mathbf{J} = 0. \tag{6}$$

Ludford (1961) has rewritten these equations in dimensionless form. By use of (3), (1) and (5) may be written

$$\frac{1}{\rho}\nabla p + (\mathbf{q}.\nabla)\mathbf{q} = \frac{S}{\rho}(\mathbf{E}+\mathbf{q}\times\mathbf{H})\times\mathbf{H}, \tag{1a}$$

and

$$\nabla\times\mathbf{H} = R_m(\mathbf{E}+\mathbf{q}\times\mathbf{H}), \tag{5a}$$

where the fluid and magnetic properties are made dimensionless by referring them to undisturbed uniform values. Here $S = \sigma\mu^2 H_0^2 l/\rho_0 U_0$ and $R_m = \mu\sigma U_0 l$ are the magnetic force coefficient and magnetic Reynolds number, based on these uniform values of density (ρ_0), field strength (H_0), velocity (U_0) and a representative length l. The units of p and E are $\rho_0 U_0^2$ and $\mu U_0 H_0$, respectively.

The following analysis is concerned with small perturbations of the uniform flow. If the current density is also restricted to a small value $\mathbf{j}$, and the magnetic Reynolds number is small, it follows from (5*a*) that $\nabla\times\mathbf{H}$ is of second-order smallness, disturbances in the field may be ignored, and the field strength $\mathbf{H}$ may be taken as the uniform applied field $\mathbf{H}_0$. An electric field $\mathbf{E} = -\mu(\mathbf{U}_0\times\mathbf{H}_0)$

must be applied to restrict the current density to the small value $\mathbf{j}$. If S is of order unity, the magnetic force term must be retained in equation (1*a*).

With these assumptions and the further assumption that the permeability of the fluid and its boundaries is that of free space (μ_0), equations (1), (2), (3) become

$$\frac{1}{\rho}\nabla p + (\mathbf{U}_0 . \nabla)\,\mathbf{c} = \frac{\mathbf{j}\times\mathbf{B}_0}{\rho}, \tag{1b}$$

$$(\nabla\rho').\mathbf{U}_0 + \rho_0(\nabla . \mathbf{c}) = 0, \tag{2b}$$

$$\mathbf{j} = \sigma(\mathbf{c}\times\mathbf{B}_0), \tag{3b}$$

where $\mathbf{c}$ is the perturbation velocity vector, ρ' is the perturbation density, and $\mathbf{B}_0 = \mu_0\mathbf{H}_0$.

Maxwell's equations are not needed, as neither electric nor magnetic fields are disturbed.

3. Incompressible flow

For incompressible flow the pressure term is eliminated by taking the curl of equation (1*a*), thus

$$(\mathbf{U}_0 . \nabla)\,\boldsymbol{\omega} = \frac{\nabla\times(\mathbf{j}\times\mathbf{B}_0)}{\rho_0} = -\frac{(\mathbf{B}_0 . \nabla)\,\mathbf{j}}{\rho_0}, \tag{7}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{c}$ is the perturbation vorticity.

Examples of two-dimensional perturbations of a uniform flow U_0 in the x-direction are now considered. Such flows may be produced if there is some small obstruction placed within the flow or at a wall. The perturbation velocities in the (x, y)-plane are u, v, and the continuity equation enables a perturbation stream function Ψ to be defined such that

$$u = \frac{\partial\Psi}{\partial y}, \quad v = -\frac{\partial\Psi}{\partial x}. \tag{8}$$

The perturbation vorticity $\zeta = \partial v/\partial x - \partial u/\partial y$ and the perturbation current j_z are in the z-direction.

(i) With a uniform 'aligned' flux $B_0 = B_x$, the electric field E_z must be zero for small current density j_z. Equations (7) and (3*b*) give

$$\frac{\partial\zeta}{\partial x} = -\frac{\sigma B_x^2}{\rho_0 U_0}\frac{\partial v}{\partial x}. \tag{9}$$

If there is no vorticity at a section where the velocity v is zero (for example, far upstream of a small obstacle placed in the flow) then equation (9) may be integrated to give

$$\zeta = -\frac{\sigma B_x^2 v}{\rho_0 U_0}, \tag{10}$$

or in terms of the stream function

$$\nabla^2\Psi = -\frac{S_x}{l}\frac{\partial\Psi}{\partial x}, \tag{11}$$

where $S_x = \sigma B_x^2 l/\rho_0 U_0$ is the magnetic force coefficient based on a representative length l.

If the applied field is not quite uniform, say $\mathbf{B}_x + \mathbf{b}(x, y)$, then the modified form of the stream function equation is

$$\nabla^2\Psi + \frac{S_x}{l}\frac{\partial \Psi}{\partial x} = -\frac{S_x b_y U_0}{l B_x} \tag{12}$$

if $\zeta = 0$, $b_y = 0$ where $v = 0$.

(ii) With a 'crossed' field $B_0 = B_y$, an electric field $E_z = -U_0 B_y$ is required to restrict the current density to small values. Equations (7) and (3b) give

$$\frac{\partial \zeta}{\partial x} = \frac{\sigma B_y^2}{\rho_0 U_0}\frac{\partial u}{\partial y}. \tag{13}$$

This equation cannot be integrated directly. It may be written in terms of the stream function

$$\frac{\partial^3 \Psi}{\partial x^3} + \frac{\partial^3 \Psi}{\partial x \partial y^2} + \frac{S_y}{l}\frac{\partial^2 \Psi}{\partial y^2} = 0, \tag{14}$$

where $S_y = \sigma B_y^2 l / \rho_0 U_0$.

Solutions of these equations for the stream function are next considered.

(a) Solutions for the incompressible flow through an aligned field

Three solutions of equation (11) have been obtained, together with a numerical solution of the equation (12).

(i) If the flow is restricted between parallel walls at $y = 0, l$, a solution of equation (11) is

$$\Psi = \sum_n A_n \sin \frac{n\pi y}{l} e^{k_n x}, \tag{15}$$

where, for $x < 0$,

$$k_n = \frac{n\pi}{l}\left[\sqrt{\left\{1 + \left(\frac{S_x}{2n\pi}\right)^2\right\}} - \frac{S_x}{2n\pi}\right], \tag{16}$$

and for $x > 0$,

$$k_n = -\frac{n\pi}{l}\left[\sqrt{\left\{1 + \left(\frac{S_x}{2n\pi}\right)^2\right\}} + \frac{S_x}{2n\pi}\right]. \tag{17}$$

A small body placed in the stream at $x = 0$ might produce such perturbations far from the body. Alternatively a sudden transverse deflexion of the stream produced by a cascade at $x = 0$ would decay at a rate governed by equations (15) to (17). Perturbations decay more rapidly downstream than upstream.*

This asymmetry may also be illustrated by considering the equation for the stream function (11). If a solution is written in the form

$$\Psi = e^{-S_x x/2l}\,\Phi(x, y), \tag{18}$$

then it follows from (11) that

$$\nabla^2\Phi = \frac{S_x^2}{4l^2}\Phi. \tag{19}$$

* If no field is applied the perturbations decay symmetrically about $x = 0$, as $\exp(\pm n\pi x/l)$. Glazebrook (1910) has derived potential-flow solutions for the flow past oval obstacles when the flow is restricted by parallel walls. It may be shown that Glazebrook's solution, for a small obstacle, gives perturbations decaying as $\exp(\pm n\pi x/l)$ upstream and downstream.

In the flow past a small obstacle located at $x = 0$, the solution for Φ will be symmetrical about the origin, but that for Ψ shows that perturbations of the uniform flow are greater upstream than downstream.

(ii) Another solution of equation (11) is

$$\Psi = \Psi_0 e^{-S_x x/l}, \tag{20}$$

which is true even without linearization. It follows that

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = 0,$$

and so the flow is one of constant pressure. Such a flow may be produced if the stream is suddenly deflected at $x = 0$; the flow returns exponentially to the x-direction and the streamlines are 'translates' in the y-direction.

(iii) If the flow is semi-infinite ($0 < y < \infty$) and the boundary at $y \to 0$ is cosinusoidal in shape and of wavelength $2l$ (i.e. $y = \epsilon \cos \pi x/l$), then the v component of velocity at this wall is

$$v = -\frac{U_0 \epsilon \pi}{l} \sin \frac{\pi x}{l},$$

and the solution for the stream function of the flow is

$$\Psi = -U_0 \epsilon \cos \frac{\pi}{l}(x + y \tan \phi) \exp\left[-\frac{\pi y \sec \phi}{l}\right], \tag{21}$$

where

$$\tan \phi \sec \phi = S_x/2\pi,$$

$$\phi = \sin^{-1}\left[\sqrt{\left\{1 + \left(\frac{\pi}{S_x}\right)^2\right\}} - \frac{\pi}{S_x}\right]. \tag{22}$$

Perturbations on the uniform flow are sinusoidal but decay exponentially along a line at an angle ϕ to the y-axis. As $S_x \to 0$, we have $\phi \to 0$, and as $S_x \to \infty$, we have $\phi \to \frac{1}{2}\pi$.

Equation (21) may be compared with the solution given by Liepmann & Roshko (1957) for flow past a wavy wall in the absence of magnetic fields. In the incompressible non-conducting case, perturbations decay exponentially with y normal to the wall, and the flow is irrotational. In the incompressible magnetohydrodynamic case, the perturbations are transmitted upstream, and there is vorticity distributed through the flow field.

The analysis given by Williams (1960) provides a link between the solution of equation (21) for small magnetic Reynolds number and the solution of Sears & Resler (1959) for infinite magnetic Reynolds number.

Williams' solution for the perturbation velocity vector in the flow past a wave-shaped wall is

$$\mathbf{c} = \frac{U_0}{B_x}\left[\nabla(P\, e^{\pi(ix-y)/l}) + \frac{1}{m^2} \nabla \times (\mathbf{k} Q\, e^{(i\pi x/l) - \beta y})\right], \tag{23}$$

where
$$P = \frac{iB_x \epsilon m^2(\beta+\pi/l)}{[2\pi/l - m^2(\beta+\pi/l)]},$$
$$Q = \frac{2i\pi/l P}{\beta+\pi/l},$$
$$\beta^2 = \pi^2/l^2\left[1+\frac{iS_x}{\pi}(m^2-1)\right],$$
$$m^2 = U_0^2\rho_0/B_x^2,$$

and $\mathbf{k}$ is the unit vector in the z-direction.

For small magnetic Reynolds number and finite S_x, m^2 is small and

$$P = \frac{iB_x \epsilon m^2}{2\pi/l}(\beta+\pi/l), \quad Q = -B_x \epsilon m^2, \quad \beta^2 = \frac{\pi^2}{l^2}\left(1-\frac{iS_x}{l}\right).$$

Hence
$$\mathbf{c} = -U_0 \epsilon \nabla \times (\mathbf{k}\, e^{i\pi x/l - \beta y}), \tag{24}$$

and
$$v = \frac{U_0 \epsilon i \pi}{l} e^{i\pi x/l - \beta y}. \tag{25}$$

This may be shown to be the same as the solution obtained above. The perturbation in the flux strength,

$$b = \nabla(P e^{\pi/l(x-iy)}) + \nabla \times (\mathbf{k} Q\, e^{i\pi x/l - \beta y}), \tag{26}$$

may be shown to be of second-order smallness for small R_m.

Williams has shown that equation (26) reduces to the result given by Resler & Sears for infinite conductivity.

(iv) The application of the slightly non-uniform flux $\mathbf{B}_x + \mathbf{b}(x, y)$ will cause perturbations of a uniform flow; the field produced by a short solenoid, for example, will act as a form of magnetic nozzle.

A relaxation solution of equation (12) has been obtained for the case in which the applied field (and flux) is made up of two components:

(*a*) a flux b_x, b_y due to two current-carrying conductors at $x = 0$, $y = \pm\frac{1}{2}l$, the flux at $x = 0$, $y = 0$ being 100 units.

(*b*) an addition flux B_x of 1000 units.

The maximum flux b_y in the duct is then approximately 100 units and the total flux varies from 1000 to 1100 units. In the relaxation solution a constant value of $S_x = 16$ (based on the flux B_x) was assumed.

The perturbation stream function obtained from this approximate solution is shown in figure 1. The maximum displacement occurs near the walls of the duct, and perturbations decay more rapidly downstream than upstream. The displacement of the streamlines is very small because of the strong axial field (the maximum value of the perturbation stream function is approximately 6 when the stream function of the uniform flow varies from zero at one wall of the duct to 750 at the other). The maximum streamline displacement is therefore less than 1 % of the duct width.

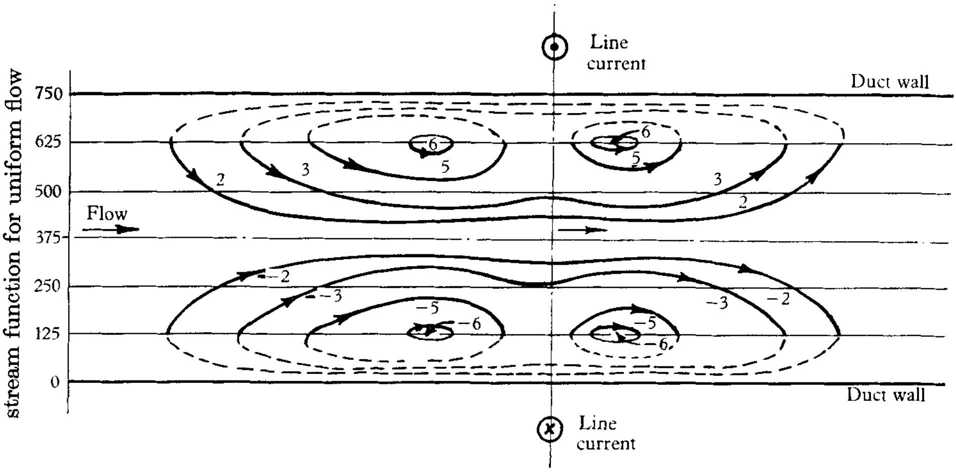


FIGURE 1. Flow through magnetic nozzle.

(b) Solutions for the incompressible flow through a crossed field

Two solutions of equation (14) have been obtained.

(i) With the flow restricted between parallel walls at $y = 0, l$, solutions for Ψ' are

$$\Psi' = \sum_n (A_n e^{\alpha_n x} + B_n e^{-\beta_n x} + C_n e^{-\gamma_n x}) \sin \frac{n\pi y}{l} \quad \text{for} \quad S_y < \frac{2n\pi}{3\sqrt{3}}, \tag{27}$$

where α_n, $-\beta_n$, $-\gamma_n$ are the real roots of the equation

$$k_n^3 - \left(\frac{n\pi}{l}\right)^2 k_n - S_y \left(\frac{n\pi}{l}\right)^3 = 0, \tag{28}$$

and $$\Psi' = \Sigma (A_n e^{\alpha_n x} + H_n e^{-\frac{1}{2}\alpha_n x}(\cos q_n x + \delta_n)) \sin \frac{n\pi y}{l} \quad \text{for} \quad S_y > \frac{2n\pi}{3\sqrt{3}}, \tag{29}$$

where α_n, $-\frac{1}{2}\alpha_n \pm i q_n$ are roots of equation (28).

Values of the first roots for various S_y are given in table 1.

S_y/π	$\alpha_1 l/\pi$	$\beta_1 l/\pi$	$\gamma_1 l/\pi$	$q_1 l/\pi$
0	1	0	1	—
$\frac{2}{3}\sqrt{3}$	$2/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$	—
$\frac{15}{8}$	$\frac{3}{2}$	—	—	$\sqrt{(11)}/4$
6	2	—	—	$\sqrt{2}$

TABLE 1

For $S_y > 2\pi/3\sqrt{3}$, damped standing waves appear downstream in any non-zero solution (provided it is not composed only of high-n normal components). Analogous solutions are described by Squire (1956) for the swirling flow past a small obstacle, although in this case the standing waves are undamped in inviscid flow.

(ii) The solution

$$\Psi' = -U_0 \epsilon \cos(\pi/l)(x + y \tan\psi) e^{-p\pi y/l} \tag{30}$$

describes the flow plast a wave-shaped wall.

Here $$p = \frac{1}{2^{\frac{1}{2}}}\frac{(1+\gamma)^{\frac{1}{2}}}{\gamma}, \tag{31}$$

and $$\tan\psi = -\frac{S_y}{2^{\frac{1}{2}}\pi}\frac{1}{\gamma(1+\gamma)^{\frac{1}{2}}}, \tag{32}$$

where $$\gamma = \left[1+\left(\frac{S_y}{\pi}\right)^2\right]^{\frac{1}{2}}. \tag{33}$$

Perturbations are transmitted downstream but decay exponentially away from the wall.

The link between this solution and that of Resler & Sears for infinite conductivity may also be made through Williams's analysis.

(iii) No constant-pressure solution similar to that of § 3 (a) exists, since here if p is constant $\partial v/\partial x = 0$ and hence $v = f(y)$. Substitution into the linearized equations of motion shows that $f(y)$ must be constant. Thus a constant velocity perturbation v may be added to the uniform flow, but this is a trivial solution.

4. Compressible flow

For compressible flow Crocco's equation

$$\nabla h_0 - T\nabla s = \mathbf{U}_0 \times \boldsymbol{\omega} + \mathbf{j}\times \mathbf{B}_0/\rho \tag{34}$$

is a more convenient starting point than the Euler momentum equation. Here h_0 is the stagnation enthalpy, T the temperature and s the entropy of the fluid. In addition the energy equation may be written

$$\rho\frac{Dh_0}{Dt} = \mathbf{E}\,.\,\mathbf{j}, \tag{35}$$

and the equation for entropy production is

$$T\frac{Ds}{Dt} = \frac{j^2}{\rho\sigma}. \tag{36}$$

The current density is restricted to small values by making $\mathbf{E} = -\mathbf{U}_0\times\mathbf{B}_0$.

(i) With a uniform applied flux $B_0 = B_x$ and $E_z = 0$, $Dh_0/Dt = 0$. Further

$$T\frac{Ds}{Dt} = \frac{\sigma v^2 B_x^2}{\rho_0} = U_0 S_x \frac{v^2}{l},$$

which is of second-order smallness.

If h_0 and s are constant across the stream at entry, then they are uniform throughout the field. From equation (34), we have, for the two-dimensional perturbation of a uniform flow,

$$\frac{1}{U_0}(\nabla h_0 - T\nabla s) = \zeta = -\frac{\sigma B_x^2 v}{\rho_0 U_0}, \tag{10a}$$

as for incompressible flow.

A perturbation stream function for compressible flow is defined by

$$\frac{\partial\Psi}{\partial y} = \rho_0 u + \rho' U_0, \quad \frac{\partial\Psi}{\partial x} = -\rho_0 v. \tag{37}$$

Since the flow is isentropic, the second Euler equation may be written

$$-\frac{a_0^2}{\rho_0}\frac{\partial \rho'}{\partial y} = U_0\frac{\partial v}{\partial x}+\frac{\sigma v B_x^2}{\rho_0}, \tag{38}$$

where $a_0^2 = (\partial p/\partial \rho)_s$ is the square of the speed of sound of the undisturbed flow.

Using equations (10*a*), (37) and (38), one obtains an equation for the stream function

$$\frac{\partial^2 \Psi}{\partial x^2}+\frac{1}{(1-M_x^2)}\frac{\partial^2 \Psi}{\partial y^2}+\frac{S_x}{l}\frac{\partial \Psi}{\partial x} = 0, \tag{39}$$

where M_x is the Mach number of the undisturbed stream.

(ii) With a uniform applied flux $B_0 = B_y$, an electric field $E_z = -U_0 B_y$ is again required to limit the current flow to small values and

$$\frac{Dh_0}{Dt} = U_0\frac{\partial h_0}{\partial x} = \frac{\sigma U_0 u B_y^2}{\rho} = \frac{S_y U_0^2 u}{l},$$

which is of first-order smallness.

Also

$$T\frac{Ds}{Dt} = \frac{\sigma u^2 B_y^2}{\rho} = \frac{S_y U_0 u^2}{l},$$

which is of second-order smallness. Hence the flow is one of isentropic work addition and subtraction.

Taking the curl of equation (34), one obtains the equation for the change of vorticity ζ, thus

$$\frac{\partial \zeta}{\partial x} = \frac{\sigma B_y^2}{\rho_0 U_0}\frac{\partial u}{\partial y}, \tag{13a}$$

as in incompressible flow.

The equation for the stream function is then

$$(1-M_x^2)\frac{\partial^3 \Psi}{\partial x^3}+\frac{\partial^3 \Psi}{\partial x \partial y^2}+\frac{S_y}{l}\frac{\partial^2 \Psi}{\partial y^2}-\frac{S_y M_x^2}{l}\frac{\partial^2 \Psi}{\partial x^2} = 0. \tag{40}$$

(*a*) *Solutions for compressible flow through an aligned field*

Equation (39) may be transformed into the corresponding incompressible equation (11) by a change in scale of the y-axis according to $y' = y(1-M_x^2)^{\frac{1}{2}}$, and the solutions of §3*a* are then valid in an (x, y')-plane, for subsonic flow.

In supersonic flow, equation (39) becomes

$$\frac{\partial^2 \Psi}{\partial x^2}-\frac{1}{\lambda^2}\frac{\partial^2 \Psi}{\partial y^2}+\frac{S_x}{l}\frac{\partial \Psi}{\partial x} = 0, \tag{41}$$

where $\lambda^2 = M_x^2-1 > 0$. A solution for the supersonic flow past a wave-shaped wall is

$$\Psi = -U_0 \epsilon \cos\frac{\pi}{l}(x+y\tan\phi')\, e^{-k'\pi y/l}, \tag{42}$$

where

$$\tan\phi' = -\frac{\lambda}{2^{\frac{1}{2}}}(\delta+1)^{\frac{1}{2}},$$

$$k' = \frac{\lambda}{2^{\frac{1}{2}}}(\delta-1)^{\frac{1}{2}},$$

and

$$\delta = \left[1+\left(\frac{S_x}{\pi}\right)^2\right]^{\frac{1}{2}}.$$

As $S_x \to 0$, we have $\delta \to 1$, $k' \to 0$ and $\tan\phi' \to -\lambda$. The solution then approaches that for the normal supersonic flow past a wavy wall, in which Mach waves are transmitted downstream at an angle $\tan^{-1}(M_x^2-1)^{\frac{1}{2}}$ to the normal. With increasing S_x the shape of the wall is transmitted further back in the flow, but perturbations decay exponentially away from the wall.

(b) Solutions for the compressible flow through a crossed field

Compressible flow solutions similar to those of § 3(*b*) may be obtained.

(i) With $\Psi = \Sigma A_n \sin(n\pi y/l)\, e^{k_n x}$, the value of k_n is again given by a cubic:

$$(1-M_x^2)\,k_n^3 - k_n\left(\frac{n\pi}{l}\right)^2 - \frac{S_y M_x^2 k_n^2}{l} - S_y\left(\frac{n\pi}{l}\right)^2 = 0. \qquad (43)$$

With S_y zero and

$$M_x < 1, \quad \text{we get} \quad k_n = \pm\frac{n\pi/l}{(1-M_x^2)^{\frac{1}{2}}},$$

the expected ordinary subsonic solution. With M_x zero, the cubic equation becomes equation (28).

The critical value of S_y that is required to form standing waves is now a function of the Mach number. For example, with $M_x = 2^{-\frac{1}{2}}$, we have

$$k_n^3 - \frac{S_y}{l}k_n^2 - 2\left(\frac{n\pi}{l}\right)^2 k_n - \frac{2S_y}{l}\left(\frac{n\pi}{l}\right)^2 = 0,$$

and with $S_y = 2\pi/3$,

$$k_1 = 2\pi/l, \quad (-\tfrac{2}{3} \pm i2^{\frac{1}{2}})\,\pi/l.$$

(ii) The solution to the flow past a wave-shaped wall is

$$\Psi = -U_0 \epsilon \cos\frac{\pi}{l}(x + y\tan\psi')\, e^{-p'\pi y/l}, \qquad (44)$$

where p' and $\tan\psi'$ are given by the equations

$$\left.\begin{aligned} p'^2 - \frac{2S_y p'}{\pi}\tan\psi' + (M_x^2 - \sec^2\psi') &= 0,\\ p'^2 + \frac{2S_y p'}{\pi}\tan\psi' + (M_x^2 - \tan^2\psi') &= 0.\end{aligned}\right\} \qquad (45)$$

5. The method of characteristics in supersonic flow through an aligned field

Hains *et al.* (1960) have given complex equations for the characteristic directions in supersonic flow at low R_m, and for the compatibility equations along these characteristics. Dr M. D. Cowley (1961) has pointed out to the author that simple characteristic relations could be derived from the equations for a small perturbation of the uniform flow through an aligned field.

Combining the momentum equation in the x-direction with the continuity equation, we get

$$\frac{\partial u}{\partial x}(M_x^2 - 1) = \frac{\partial v}{\partial y}, \qquad (46)$$

and the vorticity is

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\sigma B_x^2 v}{\rho_0 U_0}. \tag{10b}$$

Using the Prandtl–Meyer function $d\nu = U_0^{-1} \cot\mu\, du$, where $\tan\mu = (M_x^2 - 1)^{\frac{1}{2}}$, and writing $\theta = v/U_0$, where θ is the inclination of the flow to the x-axis, we obtain from equations (46) and (10*b*)

$$\frac{\partial \nu}{\partial x} - \tan\mu \frac{\partial \theta}{\partial y} = 0,$$

$$\tan\mu \frac{\partial \nu}{\partial y} - \frac{\partial \theta}{\partial x} = \frac{S_x \theta}{l}.$$

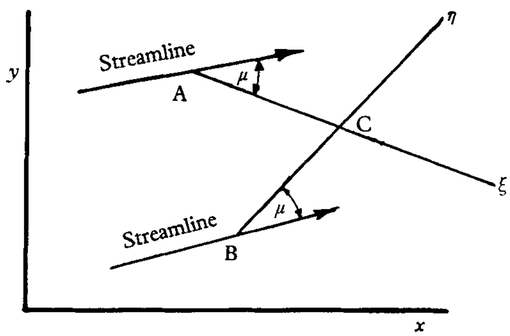


FIGURE 2. Characteristics for flow through aligned field.

If η and ξ are the characteristic directions, inclined at the Mach angle to the streamline (or to the x-axis in a perturbation flow), these equations may be transformed into

$$\sec\mu \frac{\partial}{\partial \eta}(\nu - \theta) = \frac{S_x \theta}{l},$$

$$\sec\mu \frac{\partial}{\partial \xi}(\nu + \theta) = -\frac{S_x \theta}{l}. \tag{47}$$

Along the η-characteristic,

$$\Delta(\nu - \theta) = \frac{S_x \theta}{l} \Delta\eta \cos\mu = \frac{S_x \theta}{l} \Delta x.$$

Along the ξ characteristic, $\Delta(\nu + \theta) = -S_x \theta \Delta x / l$. With S_x known, a characteristic net may be calculated for supersonic flows. From two points A and B (figure 2) characteristics may be drawn to intersect at C, and (ν, θ) at C calculated by trial and error. Three examples of this method of characteristics are given below.

(i) The supersonic flow past a sinusoidal wall has been calculated by the method of characteristics for comparison with the solution given in §4(*a*). The velocity perturbation u is obtained from the solution for Ψ as follows:

$$u = -\frac{U_0 \epsilon \pi}{2^{\frac{1}{2}} l \lambda} \left[-(\delta + 1)^{\frac{1}{2}} \sin\frac{\pi}{l}(x + y\tan\phi') + (\delta - 1)^{\frac{1}{2}} \cos\frac{\pi}{l}(x + y\tan\phi') \right] e^{-k'\pi y/l}. \tag{48}$$

The local Mach number is

$$M = M_x[1 + 2u/U_0]^{\frac{1}{2}}. \tag{49}$$

An example has been worked for $S_x = 8$, $M_x = 2\frac{1}{2}$, $\epsilon/l = \frac{1}{40}$. For these conditions we get from equations (42) $k' = 0{\cdot}932$ and $\phi' = 53^\circ\,50'$. Along a line* drawn at $53^\circ\,50'$ to the normal from the crest of the wave, $\theta = 0$ and M, ν may

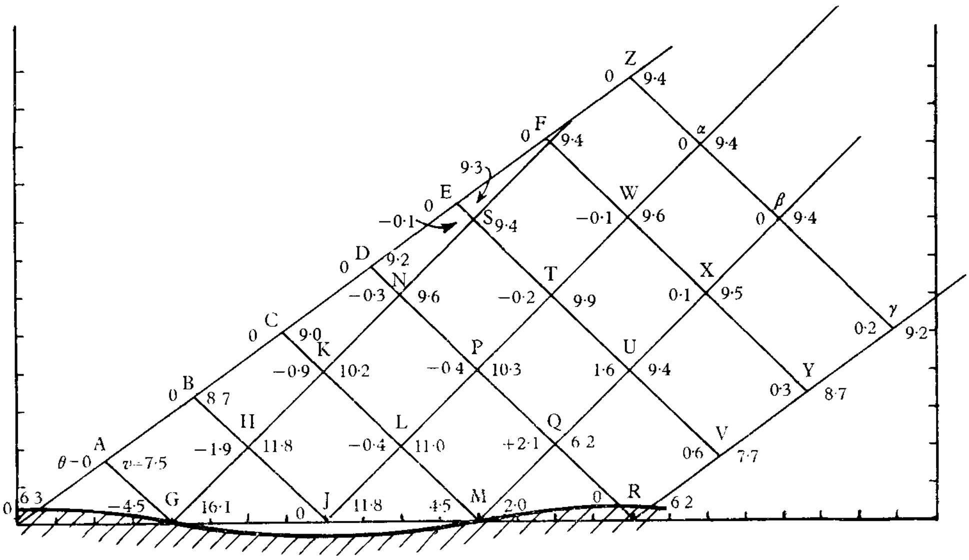


FIGURE 3. Characteristic solution for supersonic flow past wave-shaped wall.

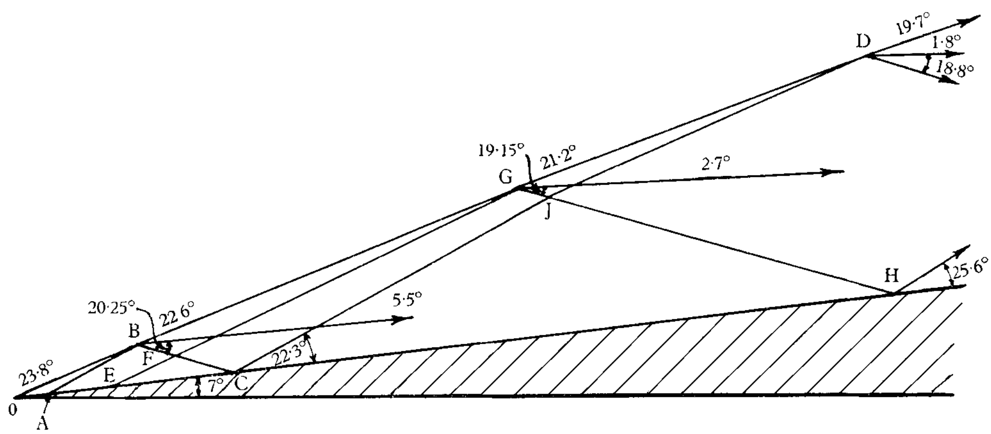


FIGURE 4. Characteristic solution for supersonic flow past wedge.

be calculated from the above equations. The characteristics were drawn at 45°, as shown in figure 3, and the values of ν and θ calculated at the next 'line of crests'. The calculations show that the distribution of ν, θ along this line is closely that assumed along the starting line for the calculations, as expected.

* Note that this line is not a characteristic.

(ii) A second calculation has been made for the supersonic flow past a thin wedge. Up to the attached shock wave starting from the nose of the wedge there is no deviation of the flow from the x-direction.

The shock wave near the wedge is inclinded at the angle given by ordinary gas dynamics. A calculation of the locus of the shock wave away from the nose has been made for an upstream Mach number of 3·2 and a wedge angle of 7° and is shown in figure 4. A characteristic line from the wedge surface behind the shock is drawn to intersect the shock wave. At the intersection, $\nu+\theta$ (behind

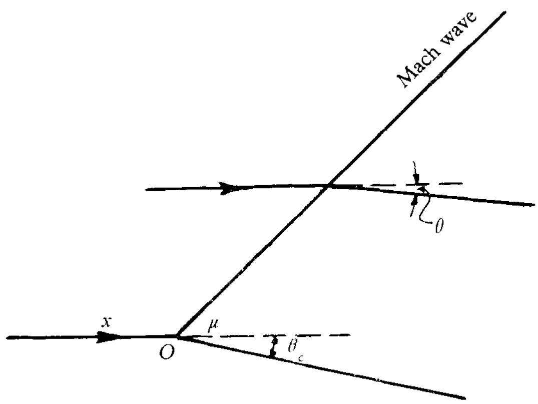


FIGURE 5. Flow past a corner.

the shock) is obtained from the characteristic equation. The shock inclination to give this value of $\nu+\theta$ is obtained by trial and error from gas tables. The solution is then carried forward for the whole area behind the shock wave.

The example (calculated by Mr N. R. Jones) shows that the wave bends over towards the wedge. A finer characteristic net than that shown in figure 4 can be used by starting other characteristics from the shock wave itself.

(iii) Dr M. D. Cowley (1961) has considered the case of an expansion at a corner (figure 5). Across the wavelet starting from the corner,

$$dx = 0 \quad \text{and} \quad d\nu = -d\theta. \tag{50}$$

Along the wavelet,

$$d\nu - d\theta = \frac{\sigma B_x^2 \, dx}{\rho_0 U_0}. \tag{51}$$

From (50) and (51) one gets

$$\frac{d\theta}{dx} = -\frac{\sigma B_x^2 \theta}{2\rho_0 U_0}, \quad \theta = \theta_c e^{-S_x x/2l}, \tag{52}$$

where subscript c refers to conditions at the corner.

The flow through a slightly under-expanded nozzle has been calculated using this analysis and the method of characteristics. The deflexion through the first wavelet is given by equation (52), and the bounding streamline is at constant ν (the flow is isentropic for small θ and the pressure and therefore the Mach number are constant along the boundary). Figure 6 shows the calculated position of the bounding streamline for an example in which the entry Mach number is $\sqrt{2}$ and the maximum deflexion at the nozzle outlet is 4°.

The deflexion through the wavelet decays exponentially along it and there will be little or no reflexion in a wide channel. The flow quickly opens out to the full flow area corresponding to the prescribed back pressure.

It should be noted that the turning given by equation (52) cannot be matched to the constant-pressure solution of §4 (a) because the exponential index in the equation differs by a factor of 2. The region behind the wavelet is not one of constant pressure because the deflexion is not uniform along the wavelet (the pressure change through the wavelet is $\rho_0 U_0^2(M_x^2-1)^{-\frac{1}{2}}\Delta\theta$ as in ordinary gas dynamics).

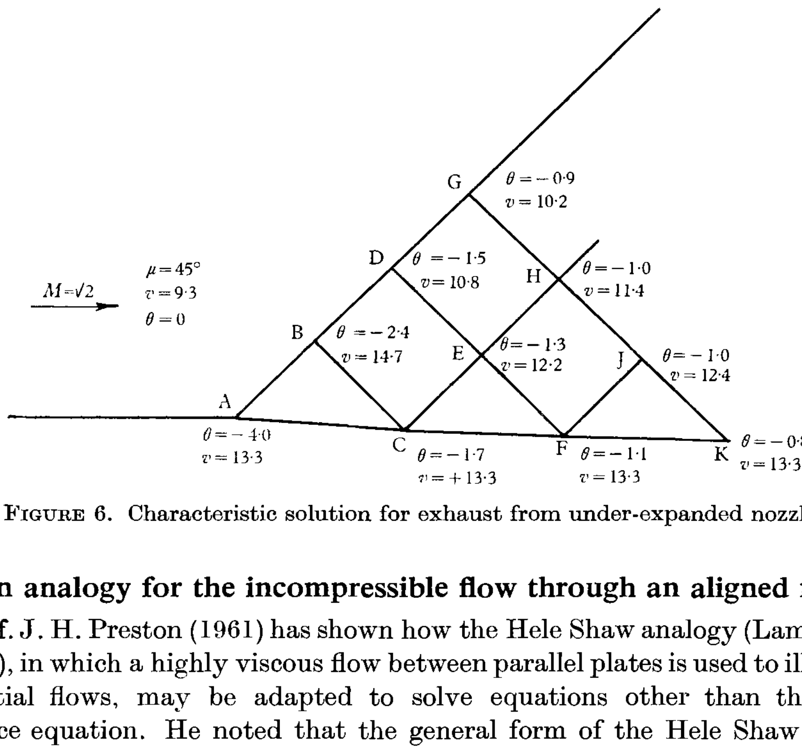


FIGURE 6. Characteristic solution for exhaust from under-expanded nozzle.

6. An analogy for the incompressible flow through an aligned field

Prof. J. H. Preston (1961) has shown how the Hele Shaw analogy (Lamb 1932, p. 582), in which a highly viscous flow between parallel plates is used to illustrate potential flows, may be adapted to solve equations other than the usual Laplace equation. He noted that the general form of the Hele Shaw stream function is

$$\nabla^2\Psi = \frac{3}{g}\frac{\partial t}{\partial x}\frac{\partial \Psi}{\partial x} + \frac{3}{g}\frac{\partial t}{\partial y}\frac{\partial \Psi}{\partial y}, \tag{53}$$

where $u = \partial\Psi/\partial y$, $v = -\partial\Psi/\partial x$, and the distance between the plates t is allowed to vary with x and y.

With an exponential variation $t = t_0 \exp(-S_x x/3l)$, the stream function form of equation (11) is obtained:

$$\nabla^2\Psi = -\frac{S_x}{l}\frac{\partial \Psi}{\partial x}.$$

Some experimental results illustrating the solutions of §3 (a) are given in figure 7 (plates 1 and 2). A variation $t = t_0 \exp(-\frac{1}{2}\pi x/l)$ was used in the experiments, where l was the width of the main stream (2·5 in.) and t was varied from

0·125 in. to 0·005 in. This corresponded to an S_x value of $\frac{3}{2}\pi = 4{\cdot}71$. The fluid used was glycerine, and the body shapes were made of wax. Dye was introduced into the flow to mark the streamlines.

Figure 7*a* (plate 1) shows the flow past a cylinder placed in the flow. The disturbances in the streamlines are more pronounced upstream than downstream, as predicted in § 3 (*a*). The same effect occurs in the flow past a plate normal to the stream (figure 7*b*).

Figure 7*c* (plate 2) shows the flow past one of two 'crests' of a wavy wall, $y = \epsilon \cos \pi x/l$. The wall shape is transmitted upstream, but decays exponentially from the wall. The angle ϕ should be $19\frac{1}{2}°$ as calculated from equation (22) and the experiment confirms that the angle is of the order of 20°.

Figure 7*d* (plate 2) shows the flow through a channel in which the wall shape is $y = y_0 \exp(-S_x x/l)$. Away from the section where the main deflexion takes place the streamlines are 'translates' in the y-direction.

7. Discussion

A variety of flows at low magnetic Reynolds number through aligned and crossed fields have been considered.

With an aligned field, perturbations are transmitted further upstream than in ordinary incompressible flow. An explanation of this effect may be given by considering the vorticity generated by a flow crossing the field lines. For example, in the flow past a wavy wall, if the streamlines are initially regarded as uninfluenced by magnetic effects (figure 8*a*), then vorticity is generated as shown in the figure. Around the circuit ABCD the circulation is counter-clockwise (positive), for the v component of velocity within the region is negative. The streamline near to the wall moves forward in incompressible magnetohydrodynamic flow (figure 8*b*), the gap g decreases in width, the velocity along DC increases and the required counter-clockwise circulation round ABCD is obtained. But in supersonic flow (figure 8*c*) a rearward movement of the streamlines widens the gap g and the velocity along DC again increases to give the required circulation.

With a crossed field, perturbations are transmitted further downstream in incompressible flow. Arguments similar to those above may be used to explain this effect. With a magnetic force coefficient greater than a certain critical value, standing waves may develop in disturbed flow past a small obstacle in the stream.

A simple method of characteristics enables supersonic flows through aligned fields H_x to be calculated. The shock wave from the leading edge of a wedge is bent back towards the main stream direction by the magnetic effects. Transverse flows across the H_x field lines, such as the expansion from an underexpanded nozzle, are quickly damped and the flows return to the x-direction.

All these flows at low magnetic Reynolds number are strongly dependent upon the magnitude of the magnetic force coefficient S.

The author wishes to thank Dr J. A. Shercliff, Dr M. D. Cowley, Dr Sheila Brenner and Mr R. A. Phillipps for useful discussion, and Mr E. T. Hobson and Mr J. Cowie for help in the experimental work (in which a modification of the original

Hele Shaw apparatus was used.) The analysis of incompressible flow given in this paper first appeared in A.R.C. Report no. 21,106 (1959).

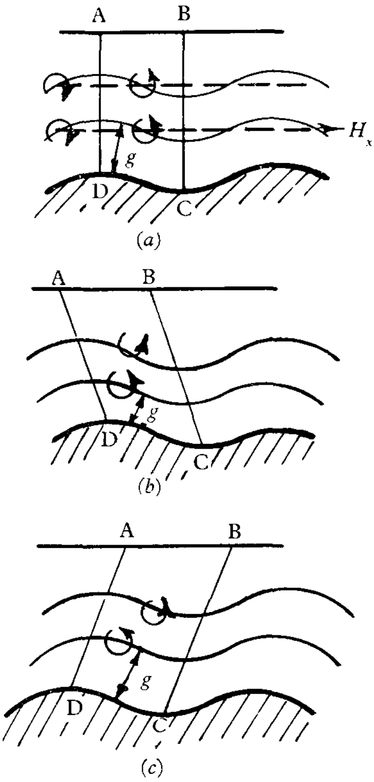


FIGURE 8. Effect of vorticity generation. (*a*) Streamlines uninfluenced by magnetic effects. (*b*) Subsonic flow. (*c*) Supersonic flow.

REFERENCES

COWLEY, M. D. 1961 Private communication.

GLAZEBROOK, R. T. 1910 *National Physical Laboratory, Collected Research*, vol. VI, p. 46.

HAINES, F. D., YOLER, Y. A. & EHLERS, E. 1960 *Dynamics of Conducting Gases*. Northwestern University Press.

HARTMANN, J. 1937 *Math. fys. Medd.* **15**, no. 6.

LAMB, H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.

LEIPMANN, H. W. & ROSHKO, A. 1957 *Elements of Gas Dynamics*. New York: Wiley.

LUDFORD, G. S. S. 1961 *J. Fluid Mech.* **10**, 141.

MCCUNE, J. E. 1960 *J. Fluid Mech.* **7**, 449.

PRESTON, J. H. 1961 Private communication.

RESLER, E. L. & MCCUNE, J. E. 1960 *J. Aero/Space Sci.* **27**, 493.

RESLER, E. L. & SEARS, W. R. 1958*a* *J. Aero. Sci.* **25**, 235.

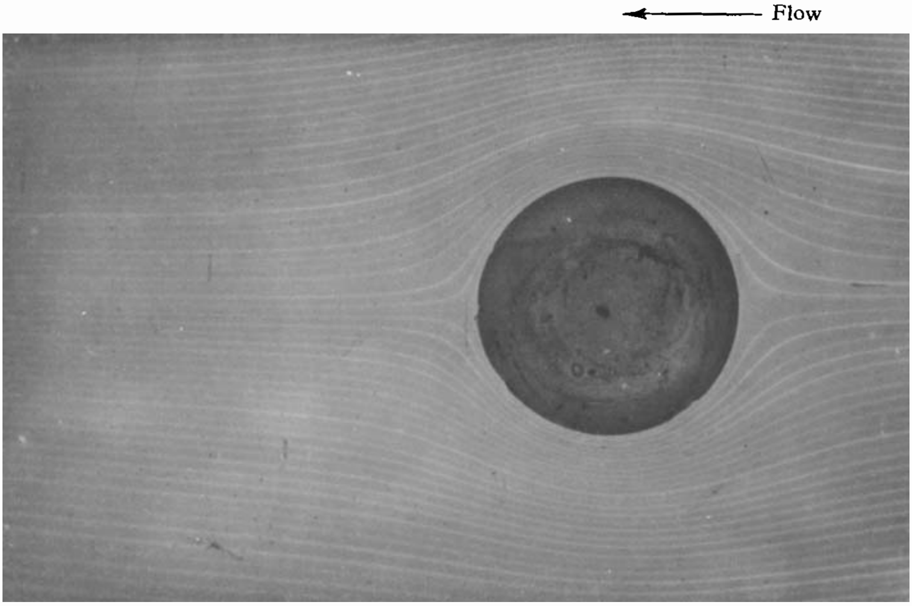
RESLER, E. L. & SEARS, W. R. 1958*b* *Z. angew. Math. Phys.*, Special no. 5–6, 509.

SEARS, W. R. & RESLER, E. L. 1959 *J. Fluid Mech.* **5**, 257.

SHERCLIFF, J. A. 1959 *J. Roy. Aero. Soc.* **63**, 518.

SQUIRE, H. B. 1956 *Rotating fluids*. Art. in *Surveys in Mechanics* (Ed. Batchelor & Davies). Cambridge University Press.

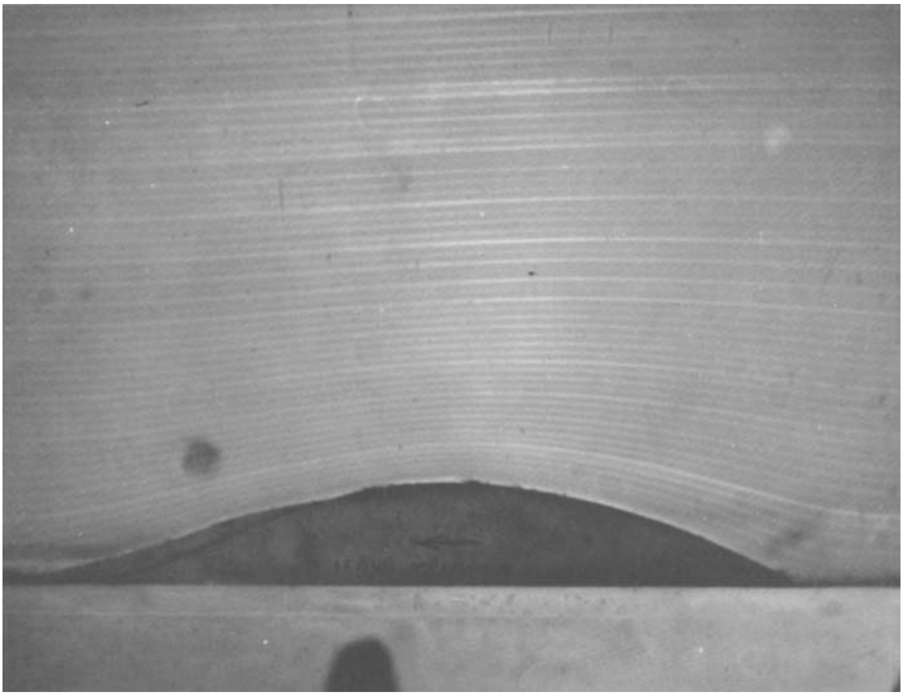
WILLIAMS, W. E. 1960 *Appl. Sci. Res.* A, **9**, 424.



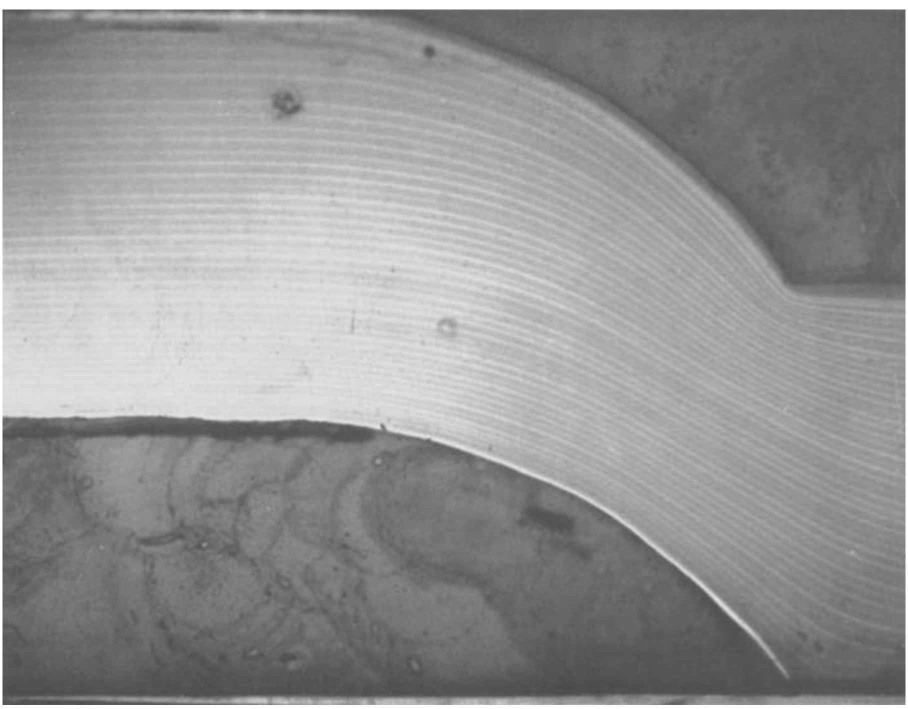
(*a*)

(*b*)

FIGURE 7. For legend see Plate 2.



(*c*) ← Flow



(*d*) ← Flow

FIGURE 7. Hele Shaw analogy for flow through aligned field. (*a*) Flow past cylinder. (*b*) Flow past normal flat plate. (*c*) Flow past wave-shaped wall. (*d*) Flow through exponential channel.

HORLOCK